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**SOME INTEGRALS INVOLVING THE (Q SUB M)
FUNCTION**

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TABLE OF CONTENTS

| | Page |
|--|------|
| LIST OF SYMBOLS | ii |
| INTRODUCTION | 1 |
| GENERAL RELATIONS FOR Q_M -FUNCTION | 1 |
| INTEGRALS | 3 |
| Bessel Functions, Exponentials, and Powers | 4 |
| Q-Function, Exponentials, and Powers | 5 |
| Q_M -Function and Powers | 5 |
| Q_M -Function, Exponentials, and Powers | 6 |
| APPENDIX — DERIVATIONS | 9 |
| REFERENCES | 13 |

LIST OF SYMBOLS

| | |
|-------------|--|
| Q_M | Q_M -function, eq. (1) |
| q_M | q_M -function, eq. (11) |
| I_k | modified Bessel function of order k |
| J_k | Bessel function of order k |
| ${}_1F_1$ | confluent hypergeometric function |
| L_k | Laguerre polynomial of degree k |
| $L_k^{(n)}$ | associated Laguerre polynomial of degree k |

SOME INTEGRALS INVOLVING THE Q_M -FUNCTION

INTRODUCTION

The detection probability for a nonfading signal present on M multiple observations and subjected to a quadratic combination of receiver matched-filter outputs is given by the Q_M -function [Ref. 1]:

$$Q_M(a, b) \equiv \int_b^{\infty} dx x \left(\frac{x}{a}\right)^{M-1} \exp\left(-\frac{x^2 + a^2}{2}\right) I_{M-1}(ax), \quad M \geq 1. \quad (1)$$

This is also called the noncentral chi-square distribution [Ref. 2]. Physically, Q_M is the probability that the random variable $\sum_{k=1}^M [(x_k + c_k)^2 + (y_k + d_k)^2]$ exceeds threshold T , where $\{x_k\}$ and $\{y_k\}$ are zero-mean independent Gaussian random variables with identical variances, σ^2 , and $\{c_k\}$ and $\{d_k\}$ are arbitrary constants. The parameters $a = \left[\sum_{k=1}^M (c_k^2 + d_k^2)/\sigma^2\right]^{1/2}$ and $b = T/\sigma$.

When the signal amplitude is subject to fading, (1) must be averaged with respect to the distribution of such fading. In this report, several integrals of the Q_M -function will be evaluated that aid in computing the performance in fading channels; direct numerical integration of the Q_M -function, which is time-consuming, is therefore circumvented. Numerical evaluation of Q_M is considered in Ref. 2.

GENERAL RELATIONS FOR Q_M -FUNCTION

Derivations of the succeeding relations are presented in the appendix:

$$Q_M(a, 0) = 1, \quad Q_M(0, \infty) = 0,$$

$$Q_M(0, b) = \exp\left(-\frac{b^2}{2}\right) \sum_{k=0}^{M-1} \frac{1}{k!} \left(\frac{b^2}{2}\right)^k, \quad Q(\infty, b) = 1. \quad (2)$$

$$\begin{aligned}
 Q_M(a, b) &= Q_{M-1}(a, b) + \left(\frac{b}{a}\right)^{M-1} \exp\left(-\frac{a^2 + b^2}{2}\right) I_{M-1}(ab) \\
 &= Q(a, b) + \exp\left(-\frac{a^2 + b^2}{2}\right) \sum_{k=1}^{M-1} \left(\frac{b}{a}\right)^k I_k(ab), \quad M \geq 2, \quad (3)
 \end{aligned}$$

where $Q = Q_1$ is the special case $M = 1$. The relation $Q_0(a, b) = 1 - Q(b, a)$ also follows from the first line of (3) and Ref. 3, (5). In the following we will generally consider the integer $M \geq 1$; however many, but not all, of the following equations apply also to $M = 0$.

$$Q_M(a, b) = \exp\left(-\frac{a^2 + b^2}{2}\right) \sum_{k=-M+1}^{\infty} \left(\frac{a}{b}\right)^k I_k(ab). \quad (4)$$

$$Q_M(a, b) = 1 - \exp\left(-\frac{a^2 + b^2}{2}\right) \sum_{k=M}^{\infty} \left(\frac{b}{a}\right)^k I_k(ab). \quad (5)$$

$$Q_M(a, b) + Q_M(b, a) = 1 + \exp\left(-\frac{a^2 + b^2}{2}\right) \sum_{k=-M+1}^{M-1} \left(\frac{b}{a}\right)^k I_k(ab). \quad (6)$$

$$Q_M(a, a) = \frac{1}{2} \left\{ 1 + \exp(-a^2) \left[I_0(a^2) + 2 \sum_{k=1}^{M-1} I_k(a^2) \right] \right\}. \quad (7)$$

$$Q_M(a, b) = \exp\left(-\frac{a^2 + b^2}{2}\right) \sum_{k=0}^{\infty} \frac{\left(\frac{a^2}{2}\right)^k}{k!} \sum_{n=0}^{k+M-1} \frac{\left(\frac{b^2}{2}\right)^n}{n!}. \quad (8)$$

$$\frac{\partial Q_M(a, b)}{\partial b} = -b \left(\frac{b}{a}\right)^{M-1} \exp\left(-\frac{a^2 + b^2}{2}\right) I_{M-1}(ab). \quad (9)$$

$$\frac{\partial Q_M(a, b)}{\partial a} = b \left(\frac{b}{a}\right)^{M-1} \exp\left(-\frac{a^2 + b^2}{2}\right) I_M(ab). \quad (10)$$

The numerical evaluation of $Q_M(a, b)$ considered in Ref. 2 is based upon a modification of (8). Notice, from (8), that $Q_M(a, b)$ is an even function of both a and b for a and b real. Also, from (1), $Q_M(a, b)$ is an analytic function of a and b for all finite complex a and b .

INTEGRALS

Numerous other integrals, in addition to those listed below, are available by the use of (3) and Ref. 3, provided the integrals on the Bessel functions in (3) can be evaluated in closed form. Rather than exhaustively list all these possibilities, some of the simpler integrals are considered and evaluated. Extensions to other integrals, which may be of interest to the reader, may profitably be pursued via (3), Ref. 3, and a good table of integrals such as found in Ref. 4.

In the integrals below, the parameters are presumed to be real and positive. The results may be generalized to negative or complex parameter values in many cases by symmetry or analytic continuation. The integrals have been checked numerically.

An integral closely related to Q_M is defined by

$$\int_b^{\infty} dx x \left(\frac{x}{a}\right)^{-M} \exp\left(-\frac{x^2 + a^2}{2}\right) I_M(ax) \equiv q_M(a, b). \quad (11)$$

Some of the properties of this function follow:

$$q_{M+1}(a, b) = q_M(a, b) - \left(\frac{a}{b}\right)^M \exp\left(-\frac{a^2 + b^2}{2}\right) I_M(ab). \quad (12)$$

$$q_0(a, b) = Q(a, b). \quad (13)$$

$$q_M(a, b) = Q(a, b) - \exp\left(-\frac{a^2 + b^2}{2}\right) \sum_{k=0}^{M-1} \left(\frac{a}{b}\right)^k I_k(ab). \quad (14)$$

$$q_M(a, b) = 1 - Q_M(b, a). \quad (15)$$

$$q_M(a, a) = \frac{1}{2} \left\{ 1 - \exp(-a^2) \left[I_0(a^2) + 2 \sum_{k=1}^{M-1} I_k(a^2) \right] \right\}. \quad (16)$$

BESSEL FUNCTIONS, EXPONENTIALS, AND POWERS

$$\int_0^{\infty} dx \exp(-p^2 x^2/2) J_{M-1}(ax) J_M(bx) = \frac{1}{a} \left(\frac{a}{b}\right)^M \left[1 - Q_M(a/p, b/p)\right]. \quad (17)$$

$$\begin{aligned} \int_0^{\infty} dx \exp(-p^2 x^2/2) J_{M-1}(ax) J_M(ax) \\ = \frac{1}{2a} \left\{1 - \exp(-a^2/p^2) \left[I_0(a^2/p^2) + 2 \sum_{k=1}^{M-1} I_k(a^2/p^2) \right] \right\}. \end{aligned} \quad (18)$$

$$\begin{aligned} \int_0^{\infty} dx \exp(-p^2 x^2/2) I_{M-1}(ax) I_M(ax) \\ = \frac{(-1)^M}{2a} \left\{1 - \exp(-a^2/p^2) \left[I_0(a^2/p^2) + 2 \sum_{k=1}^{M-1} (-1)^k I_k(a^2/p^2) \right] \right\}. \end{aligned} \quad (19)$$

In integrals (20)-(21), $s = \sqrt{p^2 - b^2}$, $u = \sqrt{a(p-s)}$, $v = \sqrt{a(p+s)}$.

$$\int_a^{\infty} dx \exp(-px) I_M(bx) = \frac{1}{sb^M} \left\{ (p-s)^M Q_M(u, v) + (p+s)^M \left[1 - Q_M(v, u)\right] \right\}, \quad p > b. \quad (20)$$

$$\int_0^a dx \exp(-px) I_M(bx) = \frac{1}{sb^M} \left\{ (p-s)^M \left[1 - Q_M(u, v)\right] - (p+s)^M \left[1 - Q_M(v, u)\right] \right\}, \quad p \neq b. \quad (21)$$

$$\int_b^{\infty} dx x^M \exp(-p^2 x^2/2) I_{M-1}(ax) = \frac{1}{a} \left(\frac{a}{p}\right)^M \exp\left(\frac{a^2}{2p^2}\right) Q_M(a/p, bp). \quad (22)$$

$$\int_b^{\infty} dx x^{-M+1} \exp(-p^2 x^2/2) I_M(ax) = \frac{1}{p^2} \left(\frac{p^2}{a}\right)^M \exp\left(\frac{a^2}{2p^2}\right) \left[1 - Q_M(bp, a/p)\right]. \quad (23)$$

Q-FUNCTION, EXPONENTIALS, AND POWERS

$$\int_0^{\infty} dx x^{2N-1} \exp\left(-\frac{p^2 x^2}{2}\right) Q(b, ax) = \frac{2^{N-1} (N-1)!}{p^{2N}} \cdot \left[1 - \frac{a^2}{p^2 + a^2} \exp\left(-\frac{b^2}{2} \frac{p^2}{p^2 + a^2}\right) \sum_{k=0}^{N-1} \left(\frac{p^2}{p^2 + a^2}\right)^k L_k\left(-\frac{b^2}{2} \frac{a^2}{p^2 + a^2}\right) \right]. \quad (24)$$

$$\int_0^{\infty} dx x^{2N-1} \exp\left(-\frac{p^2 x^2}{2}\right) Q(ax, b) = \frac{2^{N-1} (N-1)!}{p^{2N}} \frac{a^2}{p^2 + a^2} \cdot \exp\left(-\frac{b^2}{2} \frac{p^2}{p^2 + a^2}\right) \sum_{k=0}^{N-1} \epsilon_k \left(\frac{p^2}{p^2 + a^2}\right)^k L_k\left(-\frac{b^2}{2} \frac{a^2}{p^2 + a^2}\right), \quad (25)$$

where

$$\epsilon_k \equiv \begin{cases} 1, & k < N-1 \\ 1 + \frac{p^2}{a^2}, & k = N-1 \end{cases}.$$

 Q_M -FUNCTION AND POWERS

$$\int_0^{\infty} dx x^{\mu} Q_M(b, ax) = \frac{2^{\frac{1+\mu}{2}} \Gamma\left(M + \frac{1+\mu}{2}\right)}{(1+\mu) a^{1+\mu} \Gamma(M)} {}_1F_1\left(-\frac{1+\mu}{2}; M; -\frac{b^2}{2}\right), \quad \mu > -1. \quad (26)$$

$$\int_0^{\infty} dx x^{2N-1} Q_N(b, ax) = \frac{2^{N-1} (N-1)!}{a^{2N}} L_N^{(M-1)}\left(-\frac{b^2}{2}\right), \quad N \geq 1. \quad (27)$$

Q_M -FUNCTION, EXPONENTIALS, AND POWERS

$$\int_0^{\infty} dx x^{2M-1} \exp\left(-\frac{p^2 x^2}{2}\right) Q_M(ax, b) \\ = \frac{2^{M-1} (M-1)!}{p^{2M}} \exp\left(-\frac{b^2}{2} \frac{p^2}{p^2 + a^2}\right) \sum_{k=0}^{M-1} \frac{1}{k!} \left(\frac{b^2}{2} \frac{p^2}{p^2 + a^2}\right)^k. \quad (28)$$

$$\int_0^{\infty} dx x^{\rho} \exp\left(-\frac{p^2 x^2}{2}\right) Q_M(ax, b) \equiv G_M \\ = G_{M-1} + \frac{\Gamma\left(\frac{\rho+1}{2}\right) \left(\frac{b^2}{2}\right)^{M-1} \exp\left(-\frac{b^2}{2}\right)}{2(M-1)! \left(\frac{p^2 + a^2}{2}\right)^{\frac{\rho+1}{2}}} {}_1F_1\left(\frac{\rho+1}{2}; M; \frac{b^2}{2} \frac{a^2}{p^2 + a^2}\right), \quad \rho > -1. \quad (29)$$

$$\int_0^{\infty} dx x \exp\left(-\frac{p^2 x^2}{2}\right) Q_M(ax, b) = \frac{1}{p^2} \exp\left(-\frac{b^2}{2}\right) \left(\frac{p^2 + a^2}{a^2}\right)^{M-1} \\ \cdot \left[\exp\left(\frac{b^2}{2} \frac{a^2}{p^2 + a^2}\right) - \sum_{n=0}^{M-2} \frac{1}{n!} \left(\frac{b^2}{2} \frac{a^2}{p^2 + a^2}\right)^n \right] + \sum_{n=0}^{M-2} \frac{1}{n!} \left(\frac{b^2}{2}\right)^n. \quad (30)$$

$$\int_c^{\infty} dx x \exp\left(-\frac{p^2 x^2}{2}\right) Q_M(b, ax) = \frac{1}{p^2} \exp\left(-\frac{p^2 c^2}{2}\right) Q_M(b, ac) \\ - \frac{1}{p^2} \left(\frac{a^2}{p^2 + a^2}\right)^M \exp\left(-\frac{b^2}{2} \frac{p^2}{p^2 + a^2}\right) Q_M\left(\frac{ab}{\sqrt{p^2 + a^2}}, c\sqrt{p^2 + a^2}\right). \quad (31)$$

$$\begin{aligned}
\int_c^\infty dx \, x \exp(-p^2 x^2/2) Q_M(ax, b) &= \frac{1}{p} \exp(-p^2 c^2/2) Q_M(ac, b) \\
&+ \frac{1}{p} \left(\frac{p^2 + a^2}{a^2} \right)^{M-1} \exp\left(-\frac{b^2}{2} \frac{p^2}{p^2 + a^2}\right) \left[1 - Q_M\left(c\sqrt{p^2 + a^2}, \frac{ab}{\sqrt{p^2 + a^2}}\right) \right].
\end{aligned}
\tag{32}$$

$$\begin{aligned}
\int_0^\infty dx \, x^M \exp(-p^2 x^2/2) I_{M-1}(cx) Q_M(ax, b) \\
= \frac{1}{c} \left(\frac{c}{p} \right)^M \exp\left(\frac{c^2}{2p^2}\right) Q_M\left(\frac{ac}{p\sqrt{p^2 + a^2}}, \frac{bp}{\sqrt{p^2 + a^2}}\right).
\end{aligned}
\tag{33}$$

Appendix

DERIVATIONS

(2)*: The first two relations in (2) follow directly from definition (1); Ref. 4, 6.631 4; and Ref. 5, 9.6.3. The third relation results if we employ Ref. 5, 9.6.7, and the integral

$$\int_c^\infty dx \frac{x^{M-1} e^{-x}}{(M-1)!} = e^{-c} \sum_{k=0}^{M-1} \frac{c^k}{k!}. \quad (A-1)$$

The fourth relation follows by use of

$$Q_M(a, b) = 1 - \int_0^b dx x \left(\frac{x}{a}\right)^{M-1} \exp\left(-\frac{x^2 + a^2}{2}\right) I_{M-1}(ax) \quad (A-2)$$

and as a result of the asymptotic behavior of I_{M-1} (Ref. 5, 9.7.1).

(3): Integrate by parts with

$$u = \left(\frac{x}{a}\right)^{M-1} I_{M-1}(ax), \quad dv = dx x \exp\left(-\frac{x^2 + a^2}{2}\right), \quad (A-3)$$

and employ Ref. 5, 9.6.28. The second line in (3) follows by repeated application of the first line.

(4): Use (3); Ref. 3, (3); and Ref. 5, 9.6.6.

(5): Use (3) and Ref. 3, (4).

(6): Use (3); Ref. 3, (5); and Ref. 5, 9.6.6.

(7): Let $b = a$ in (6).

*The number at the beginning of each paragraph refers to the equation of that same number in the main text of this report.

(8): Expand I_{M-1} in (1) in a power series and integrate term by term using (A-1).

(9): Differentiate (1) with respect to b .

(10): Differentiate (3) with respect to a , and employ Ref. 3, (8), and Ref. 5, 9.6.28.

(11): This is a definition of $q_M(a, b)$.

(12): Integrate (11) by parts using

$$u = \left(\frac{x}{a}\right)^{-M} I_M(ax), \quad dv = dx \, x \exp\left(-\frac{x^2 + a^2}{2}\right) \quad (A-4)$$

and, from Ref. 5, 9.6.28,

$$\frac{d}{dy} \left\{ \left(\frac{y}{\alpha}\right)^{-n} I_n(\alpha y) \right\} = y \left(\frac{y}{\alpha}\right)^{-n-1} I_{n+1}(\alpha y). \quad (A-5)$$

Then use (11) again.

(13): Use definition of $Q(a, b)$ in (1).

(14): Use (12) and (13).

(15): Add (14) to the second line of (3), with a and b interchanged; use Ref. 3, (5).

(16): Set $b = a$ in (15) and use (7).

(17): From Ref. 4, 6.533 2,

$$\begin{aligned} b^M \exp\left(-\frac{a^2 + b^2}{2}\right) I_{M-1}(ab) &= b^M \int_0^\infty dt \, t \exp\left(-\frac{t^2}{2}\right) J_{M-1}(at) J_{M-1}(bt) \\ &= \frac{\partial}{\partial b} \int_0^\infty dt \exp\left(-\frac{t^2}{2}\right) J_{M-1}(at) b^M J_M(bt); \quad (A-6) \end{aligned}$$

the last step was obtained via Ref. 5, 9.1.30. Integrate (A-6) with respect to b and use (1) and (2).

(18): Set $b = a$ in (17) and use (7).

(19): Replace a by ia in (18) and use Ref. 5, 9.6.3 and 9.6.30.

(20): Utilize the method used for (24) in Ref. 3, p. 24; employ (4) and (5) to sum the series; and then use (3).

(21): Use

$$\int_0^a dx g(x) = \int_0^\infty dx g(x) - \int_a^\infty dx g(x) \quad (A-7)$$

in conjunction with (2) and (20). The special case of $p = b$ is available directly in Ref. 5, 11.3.13.

(22): Use definition (1).

(23): Employ (5) and (15).

(24): Express Q in integral form via (1) (for $M = 1$); interchange integrals; and use (A-1), Ref. 4, 6.631 1, and Ref. 5, 13.1.27 and 13.6.9. L_k is a Laguerre polynomial.

(25): Use Ref. 3, (5); (A-1); Ref. 4, 6.631 1; and Ref. 5, 13.1.27 and 13.6.9.

(26): Use (1); interchange integrals; and use Ref. 4, 6.631 1.

(27): Use (26) and Ref. 5, 13.6.9.

(28): Express Q_M in integral form via (1); interchange integrals; and use Ref. 4, 6.631 4, and (A-1).

(29): Employ the upper line of (3) and Ref. 4, 6.631 1. For p an odd integer, the recursion, (29), can be started via (Ref. 5, chap. 13):

$${}_1F_1(0;M;x) = 1,$$

$${}_1F_1(1;M;x) = (M-1)! x^{-M+1} \left[\exp(x) - \sum_{k=0}^{M-2} \frac{x^k}{k!} \right],$$

$${}_1F_1(n;M;x) = \frac{1}{n-1} \left[(M+1-n) {}_1F_1(n-2;M;x) + (2n-2-M+x) {}_1F_1(n-1;M;x) \right], \quad n \geq 2.$$

(A-8)

For p an even integer, we need ${}_1F_1\left(\frac{1}{2};M;x\right)$, ${}_1F_1\left(\frac{3}{2};M;x\right)$, and recursion.

(30): Use (1); interchange integrals; employ Ref. 4, 6.631 1, and (A-8) and (A-1); interchange the double sum; and employ

$$\sum_{k=0}^{n-1} x^k = \frac{1-x^n}{1-x}.$$

(A-9)

(31): Express Q_M in integral form via (1), interchange integrals, and use (22) twice. If we replace p by ip in (31), the integral converges for $p < a$.

(32): Integrate by parts, with

$$u = Q_M(ax, b), \quad dv = dx x \exp\left(-p \frac{x^2}{2}\right),$$

(A-10)

and use (10) and (23).

(33): Take the partial derivative of the integral with respect to b ; use (9) and Ref. 4, 6.633 2; then, utilizing the fact that the integral in (33) is zero at $b = \infty$, integrate on b and use (22).

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